

System of viscous conservation laws:

$$u_t + f(u)_x = \mu u_{xx}$$

Profile:

$$u^*(x, t) = \phi(x \Leftrightarrow st)$$

$$\mu \phi' = h(\phi) = f(\phi) \Leftrightarrow s\phi \Leftrightarrow q, \quad \phi(\pm\infty) = u_{\pm}$$

Integrated perturbation:

$$U(x \Leftrightarrow st, t) = \int_{-\infty}^x u(\xi, t) \Leftrightarrow \phi(\xi \Leftrightarrow st) d\xi$$

Integrated equation:

$$U_t + h'(\phi)U_x \Leftrightarrow \mu U_{xx} = F(\phi, U_x)$$

$$\text{where } h' = f' \Leftrightarrow sI$$

Main task: Proof

$$\|U(\cdot, T)\|_{L^2}^2 + \int_0^T \|U_x\|_{L^2}^2 dt \leq C \|U(\cdot, 0)\|_{L^2}^2$$

( $\rightsquigarrow$  stability).

# Stability of profiles of general small-amplitude Laxian shock waves (via Weighted Energy Estimates)

## 1. Viscous Shock Profiles

Hyperbolic conservation law

$$u_t + f(u)_x = \mu u_{xx} \quad x \in \mathbb{R}, u \in \mathbb{R}^n \quad (1)$$
$$u(\pm\infty, t) = u_{\pm}$$

(i.e.  $f'(u)$  IR-diagonalizable  $\forall u$ ).

Consider a traveling wave solution

$$u^*(x, t) = \phi(x \leftrightarrow st)$$

i.e.,  $\phi$  solves

$$\mu \phi' = h(\phi) = f(\phi) \leftrightarrow s\phi \leftrightarrow q \quad , \quad \phi(\pm\infty) = u_{\pm} \quad ,$$

where  $u_-$ ,  $u_+$ ,  $s$ ,  $q$  satisfy

$$f(u_-) \leftrightarrow su_- = f(u_+) \leftrightarrow su_+ = q \quad ,$$

## Definition

$u^*(x, t) = \phi(x \Leftrightarrow st)$  asymptotically stable

$:\Leftrightarrow \exists (B, \|\cdot\|_B), \beta > 0: \forall u_0 \in B, \|u_0\|_B < \beta:$

Solution  $u$  of (1) with perturbed data

(S)  $u(\cdot, 0) = \phi + u_0$  exists for all  $t > 0$  and has

$$\lim_{t \rightarrow \infty} \sup_x |u(x, t) \Leftrightarrow \phi(x \Leftrightarrow st)| = 0$$

## Theorem (Goodman, 1985)

- $\lambda$  simple eigenvalue of  $f'$
- “convex”:  $\mathbb{R} \cdot r = \ker(f' \Leftrightarrow \lambda I) \Rightarrow r \cdot \nabla \lambda \neq 0$
- $\phi$  associated with  $\lambda$ :  $\lambda(u_-) > s > \lambda(u_+)$
- small amplitude:  $|u_+ \Leftrightarrow u_-| \ll 1$

$\Rightarrow$  (S) for all  $u_0 \in L^1(\mathbb{R})$  with  $\int_{-\infty}^{\infty} u_0 dx = 0$

and  $\|U_0\|_{H^2} < \beta$  ( $U_0(x) := \int_{-\infty}^x u_0(y) dy$ )

Goal:

## Theorem 1

Same without “convex”

## 2. The Integrated Equation

Subtract the solution  $u^*(x, t) = \phi(x \leftrightarrow st)$  from a solution  $u$  (with perturbed initial data  $\phi + (U_0)_x$ ):

$$(u \leftrightarrow u^*)_t + (f(u) \leftrightarrow f(u^*))_x = \mu(u \leftrightarrow u^*)_{xx}$$

Integrated perturbation:

$$U(x \leftrightarrow st, t) = \int_{-\infty}^x u(x, t) \leftrightarrow \phi(x \leftrightarrow st) dx$$

Integrate and change to moving coordinates  $(x \leftrightarrow st, t)$ ):

$$\begin{aligned} \leftrightarrow s U_x + U_t + f(\phi + U_x) \leftrightarrow f(\phi) &= \mu U_{xx} \\ U(x, 0) = U_0(x) &= \int_{-\infty}^x u(x, 0) \leftrightarrow \phi(x) dx \end{aligned}$$

Using Taylor expansion

$$f(\phi + U_x) \leftrightarrow f(\phi) = f'(\phi)U_x \leftrightarrow F(\phi, U_x)$$

### Integrated Equation

$$\begin{aligned} U_t + h'(\phi)U_x \leftrightarrow \mu U_{xx} &= F(\phi, U_x) \quad (2) \\ U(\cdot, 0) = U_0 &= \int_{-\infty}^x u(x, 0) \leftrightarrow \phi(x) dx \end{aligned}$$

where  $h = f \leftrightarrow \text{id}$ .

### 3. Example: Scalar case, convex flux

Apply a standard energy estimate to the integrated equation

$$U_t + h'(\phi)U_x \Leftrightarrow \mu U_{xx} = F(\phi, U_x)$$

Multiply by  $U$  and integrate  $\int_{-\infty}^{\infty} dx, \int_0^T dt$ :

$$\begin{aligned} & \|U(\cdot, T)\|_{L^2}^2 + \int_0^T \int_{-\infty}^{\infty} \Leftrightarrow (h'(\phi))_x U^2 dx dt \\ & + \int_0^T \|U_x\|_{L^2}^2 dt \leq C \|U(\cdot, 0)\|_{L^2}^2 \end{aligned}$$

proceeding in a similar (but less restrictive) way with  $U_x$  and  $U_{xx}$  we have

#### Proposition 1

If  $n = 1$  (scalar equation) and  $(h'(\phi))_x < 0$  then

$$\|U(\cdot, T)\|_{H^{2,2}}^2 + \int_0^T \|U_x\|_{H^{2,2}}^2 dt \leq C \|U(\cdot, 0)\|_{H^{2,2}}^2$$

This a-priori estimate ensures the existence of  $U$  for all times and gives a decay for  $U_x$  such that Theorem 1 holds.

#### 4. Scalar case, non-convex flux (Matsumura & Nishihara 1994):

Multiply by  $U \cdot w$ ,  $w = w(x)$  and integrate  $\int_{-\infty}^{\infty} dx$ :

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} (w|U|^2) dx + \int_{-\infty}^{\infty} U w h'(\phi) U_x + \mu U w_x U_x \\ & + \mu w (U_x)^2 \Leftrightarrow U w F(\phi, U_x) dx = 0 \end{aligned}$$

Find positive weight  $w$  such that:

$$\Leftrightarrow \frac{1}{2} (w h'(\phi) + \mu w_x)_x > 0$$

Ansatz  $w(x) = \tilde{w}(\phi(x))$ :

$$\begin{aligned} & \Leftrightarrow \frac{1}{2} (w h'(\phi) + \mu w_x)_x \\ = & \Leftrightarrow \frac{1}{2} (\tilde{w}(\phi) h'(\phi) + \mu \tilde{w}'(\phi) \phi_x)_x \\ = & \Leftrightarrow \frac{1}{2} ((\tilde{w} h)'(\phi))_x = \Leftrightarrow \frac{1}{2} (\tilde{w} h)''(\phi) \phi_x \end{aligned}$$

Choose

$$\tilde{w}(u) = \Leftrightarrow \frac{(u \Leftrightarrow u_+)(u \Leftrightarrow u_-)}{h} \cdot \text{sign } \phi_x > 0$$

to obtain

$$\Leftrightarrow \frac{1}{2} (w h'(\phi) + \mu w_x)_x = |\phi_x|$$

## 5. System case, non-convex mode

Diagonalize (Goodman 1985): Matrix function

$$L(x) = \tilde{L}(\phi(x)), R(x) = \tilde{R}(\phi(x))$$

such that  $LR \equiv I$  and:

$$L(h'(\phi))R = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$$

where

$$\lambda_p = \lambda \Leftrightarrow s, \quad \lambda_i < 0 < \lambda_j \quad (i < p < j)$$

Substitute  $U =: RV$  in (2), multiply by  $V^T W L$ , integrate  $\int_{-\infty}^{\infty} dx \rightsquigarrow$

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{1}{2} \frac{\partial}{\partial t} (V^T W V) + V^T W \Lambda V_x + V^T W \Lambda L R_x V \\ & + \mu (V^T W L)_x (RV)_x \Leftrightarrow V^T W L F(\phi, (RV)_x) dx \\ & = 0 \end{aligned}$$

Choose

$$W = \text{diag}(1, \dots, 1, w, 1, \dots, 1)$$

Group terms:

$$(A1) \frac{1}{2} \frac{\partial}{\partial t} (V^T W V)$$

$$(A2) (w\lambda_p + \mu w_x) V_p (V_p)_x$$

$$(A3) \sum_{k \neq p} (\Leftrightarrow \frac{1}{2} (\lambda_k)_x + l_k (r_k)_x \lambda_k) (V_k)^2$$

$$(A4) \mu w ((V_p)_x)^2 + \mu \sum_{k \neq p} ((V_k)_x)^2$$

$$(B1) \sum_{j \neq p} (w\lambda_p + \mu w_x) l_p (r_j)_x V_p V_j$$

$$(B2) \sum_{i \neq p, i \neq j} \lambda_i l_i (r_j)_x V_i V_j$$

$$(B3) \mu V^T W L_x R V_x + \mu V_x^T W L R_x V$$

$$(B4) \mu V^T W L_x R_x V$$

$$(B5) \Leftrightarrow V^T W L F(\phi, (R V)_x)$$

## Remarks (on the diagonalization):

1.  $|L_x|, |R_x| \leq O(1)|\phi_x| \leq O(1)|u_+ \Leftrightarrow u_-|$ .
2. If  $l_k, r_k$  are left and right eigenvectors of  $h'$  ( $l_i r_j = \delta_{ij}$ ), so are  $\frac{1}{\alpha_k} l_k$  and  $\alpha_k r_k$ , i.e. there are  $n$  degrees of freedom in our choice of  $L$  and  $R$ .

For  $k \neq p$ , use  $\alpha_k$  to achieve positivity in the  $V_k$ -component:

$$(A3) = \left( \underbrace{\frac{1}{2}(\lambda_k)_x}_{|\cdot| \leq C|\phi_x|} + \underbrace{\frac{(\alpha_k)_x}{\alpha_k} \lambda_k}_{\geq 2C|\phi_x|} \right) (V_k)^2$$

by choice of  $\alpha_k$   
(as  $\lambda_k$  uniformly  
away from 0)

Not so for  $k = p$  (as  $\lambda_p$  crosses 0 along  $\phi$ ).

For  $k = p$ , use  $w$ .

## Lemma

$\epsilon := |u_- \Leftrightarrow u_+| \ll 1$  then

$\exists w : \mathbb{R} \rightarrow \mathbb{R}$  with  $\inf_x w(x), \inf_x (1/w(x)) > 0$  ( $\leadsto$   $L_2$ -norm in (A1), (A4) and estimate for (B5)) and

$$\Leftrightarrow \frac{1}{2} (w \lambda_p + \mu w_x)_x = |\phi_x|$$

( $\leadsto$  positivity of (A2))

$$|w \lambda_p + \mu w_x| \leq 4 |u_+ \Leftrightarrow u_-|$$

( $\leadsto$  estimate for (B1))

$$|\mu (w \phi_x)_x| = |(w h(\phi))_x| \leq 4 |u_+ \Leftrightarrow u_-| \cdot |\phi_x|$$

( $\leadsto$  estimate for (B3))

$$|\mu w \phi_x| = |w h(\phi)| \leq 8 |u_+ \Leftrightarrow u_-|^2$$

( $\leadsto$  estimate for (B4))

## Proof of Lemma

Analyze the differential equation

$$\mu w_x + w \lambda_p = \int_{x_0}^x |\phi_x| d\xi \quad , \quad w(0) = w_0$$

and choose the parameters  $x_0, w_0$  appropriately.

## Proof of Theorem

Apply Lemma and estimate (B1) to (B5)